Trivial Lagrangians in the Causal Approach

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Abstract

We prove the non-uniqueness theorem for the chronological products of a gauge model. We use a cohomological language where the cochains are chronological products, gauge invariance means a cocycle restriction and coboundaries are expressions producing zero sandwiched between physical states. Suppose that we have gauge invariance up to order n of the perturbation theory and we modify the first-order chronological products by a coboundary (a trivial Lagrangian). Then the chronological products up to order n get modified by a coboundary also.

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1 Introduction

The general framework of perturbation theory consists in the construction of the chronological products such that Bogoliubov axioms are verified [1], [4], [2]; for every set of Wick monomials $A_1(x_1), \dots, A_n(x_n)$ acting in some Fock space $\mathcal{H}$ one associates the operator

$$T(A_1(x_1), \dots, A_n(x_n))$$

which is a distribution-valued operators called chronological product.

The construction of the chronological products can be done recursively according to Epstein-Glaser prescription [4], [5] (which reduces the induction procedure to a distribution splitting of some distributions with causal support) or according to Stora prescription [7] (which reduces the renormalization procedure to the process of extension of distributions). These products are not uniquely defined but there are some natural limitation on the arbitrariness. If the arbitrariness does not grow with n we have a renormalizable theory.An equivalent point of view uses retarded products [11].

Gauge theories describe particles of higher spin. Usually such theories are not renormalizable. However, one can save renormalizablility using ghost fields. Such theories are defined in a Fock space $\mathcal{H}$ with indefinite metric, generated by physical and un-physical fields (called *ghost fields*). One selects the physical states assuming the existence of an operator Q called *gauge charge* which verifies $Q^2 = 0$ and such that the *physical Hilbert space* is by definition $\mathcal{H}_{\rm phys} \equiv Ker(Q)/Im(Q)$. The space $\mathcal{H}$ is endowed with a grading (usually called *ghost number*) and by construction the gauge charge is raising the ghost number of a state. Moreover, the space of Wick monomials in $\mathcal{H}$ is also endowed with a grading which follows by assigning a ghost number to every one of the free fields generating $\mathcal{H}$. The graded commutator d_Q of the gauge charge with any operator A of fixed ghost number

$$d_Q A = [Q, A] \tag{1.1}$$

is raising the ghost number by a unit. It means that d_Q is a co-chain operator in the space of Wick polynomials. From now on $[\cdot, \cdot]$ denotes the graded commutator. From $Q^2 = 0$ one derives

$$(d_Q)^2 = 0. \tag{1.2}$$

A gauge theory assumes also that there exists a Wick polynomial of null ghost number $T(x)$ called *the interaction Lagrangian* such that

$$[Q, T] = i \partial_\mu T^\mu \tag{1.3}$$

for some other Wick polynomials T^μ. This relation means that the expression T leaves invariant the physical states, at least in the adiabatic limit. Indeed, if this is true we have:

$$T(f)\ \mathcal{H}_{\rm phys}\ \subset\ \ \mathcal{H}_{\rm phys} \tag{1.4}$$

up to terms which can be made as small as desired (making the test function f flatter and flatter). We call this argument the *formal adiabatic limit*. It is a way to justify from the physical point of view relation (1.3). Otherwise, we simply have to postulate it.

In all known models one finds out that there exist a chain of Wick polynomials $T^{\mu},\ T^{\mu\nu},\ldots$ such that:

$$[Q,T] = i\partial_{\mu}T^{\mu}, \quad [Q,T^{\mu}] = i\partial_{\nu}T^{\mu\nu}, \quad [Q,T^{\mu\nu}] = i\partial_{\rho}T^{\mu\nu\rho},\ldots \tag{1.5}$$

It so happens that for all these models the expressions $T^{\mu\nu},\ T^{\mu\nu\rho},\ldots$ are completely antisymmetric in all indexes; it follows that the chain of relation stops at the step 4 (if we work in four dimensions). We can also use a compact notation T^{I} where I is a collection of indexes $I = [\nu_1,\ldots,\nu_p]$ $(p = 0,1,\ldots,)$ and the brackets emphasize the complete antisymmetry in these indexes. All these polynomials have the same canonical dimension

$$\omega(T^{I}) = \omega_0,\ \forall I \tag{1.6}$$

and because the ghost number of $T \equiv T^{\emptyset}$ is supposed null, then we also have:

$$gh(T^{I}) = |I|. \tag{1.7}$$

One can write compactly the relations (1.5) as follows:

$$d_{Q}T^{I} = i\ \partial_{\mu}T^{I\mu}. \tag{1.8}$$

For concrete models the equations (1.5) can stop earlier: for instance in the Yang-Mills case we have $T^{\mu\nu\rho} = 0$ and in the case of gravity $T^{\mu\nu\rho\sigma} = 0$.

Now we can construct the chronological products $T(T^{I_1}(x_1),\ldots,T^{I_n}(x_n))$ according to the recursive procedure. We say that the theory is gauge invariant in all orders of the perturbation theory if the following set of identities generalizing (1.8):

$$d_{Q}T(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) = i\sum_{l=1}^{n}(-1)^{s_l}\frac{\partial}{\partial x^{\mu}_{l}}T(T^{I_1}(x_1),\ldots,T^{I_l\mu}(x_l),\ldots,T^{I_n}(x_n)) \tag{1.9}$$

are true for all $n \in \mathbb{N}$ and all $I_1,\ldots,I_n$. Here we have defined

$$s_l \equiv \sum_{j=1}^{l-1}|I|_j. \tag{1.10}$$

In particular, the case $I_1 = \ldots = I_n = \emptyset$ it is sufficient for the gauge invariance of the scattering matrix, at least in the adiabatic limit: we have the same argument as for relation (1.4).

To describe this property in a cohomological framework, we consider that the chronological products are the cochains and we define for the operator δ by

$$\delta T(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) = i\sum_{l=1}^{n}(-1)^{s_l}\frac{\partial}{\partial x^{\mu}_{l}}T(T^{I_1}(x_1),\ldots,T^{I_l\mu}(x_l),\ldots,T^{I_n}(x_n)). \tag{1.11}$$

It is easy to prove that we have:

$$\delta^2 = 0 \tag{1.12}$$

and

$$[d_Q, \delta] = 0. \tag{1.13}$$

Next we define

$$s \equiv d_Q - i\delta \tag{1.14}$$

such that relation (1.9) can be rewritten as

$$sT(T^{I_1}(x_1), \ldots, T^{I_n}(x_n)) = 0. \tag{1.15}$$

We note that if we define

$$\bar{s} \equiv d_Q + i\delta \tag{1.16}$$

we have

$$s\bar{s} = 0, \qquad \bar{s}s = 0 \tag{1.17}$$

so expressions verifying the relation $sC = 0$ can be called *cocycles* and expressions of the type $\bar{s}B$ are the *coboundaries*. One can build the corresponding cohomology space in the standard way.

The use of this construction is the following. The expressions T^I are not unique. Indeed the redefinitions by a coboundary

$$T^I \rightarrow T^I + \bar{s}B^I \tag{1.18}$$

preserve the relation (1.8) and (with appropriate restrictions coming from Lorentz invariance and canonical dimension) it is the most general redefinition preserving gauge invariance (1.8). Expressions of the type $\bar{s}B$ i.e. coboundaries are trivial from the physical point of view: they give zero when restricted to the physical subspace (in the formal adiabatic limit) so they are *trivial Lagrangians.*

We are interested in the following problem. Suppose that we modify the expressions T^I by a coboundary (i.e. a trivial Lagrangian). Then in what way would be modified the chronological products in an arbitrary order n? We will prove that if we impose (1.9) for $1, 2, \ldots, n$ the modification of the chronological products in order n is also a coboundary i.e. something trivial from the physical point of view. This problem was addressed (in the causal formalism) for the first time in [3] but no complete proof is provided.

In the next Section we give the essential ingredients for a causal gauge theory. The we will prove the result announced above in Section 3.

2 Bogoliubov Axioms

Suppose that the Wick monomials $A_1, \ldots, A_n$ are self-adjoint: $A_j^\dagger = A_j,\ \forall j = 1, \ldots, n$ and of Fermi number f_i. We impose the *causality* property:

$$A_j(x)\ A_k(y) = (-1)^{f_j f_k}\ \ A_k(y)\ A_j(x) \tag{2.1}$$

for $(x-y)^2 < 0$ i.e. $x - y$ outside the causal cones (this relation is denoted by $x \sim y$).

The chronological products $T(A_1(x_1), \ldots, A_n(x_n))\quad n = 1, 2, \ldots$ are verifying the following set of axioms:

- Skew-symmetry in all arguments

$$T(\ldots, A_i(x_i), A_{i+1}(x_{i+1}), \ldots,) = (-1)^{f_i f_{i+1}} T(\ldots, A_{i+1}(x_{i+1}), A_i(x_i), \ldots) \tag{2.2}$$

- Poincaré invariance: we have a natural action of the Poincaré group in the space of Wick monomials and we impose that for all $g \in inSL(2, \mathbb{C})$ we have:

$$U_g T(A_1(x_1), \ldots, A_n(x_n)) U_g^{-1} = T(g \cdot A_1(x_1), \ldots, g \cdot A_n(x_n)) \tag{2.3}$$

  where in the right hand side we have the natural action of the Poincaré group on Wick monomials (build from Lorentz covariant free fields).

  Sometimes it is possible to supplement this axiom by other invariance properties: space and/or time inversion, charge conjugation invariance, global symmetry invariance with respect to some internal symmetry group, supersymmetry, etc.

- Causality: if $x - y$ is in the upper causal cone then we denote this relation by $x \succeq y$. Suppose that we have $x_i \succeq x_j, \quad \forall i \leq k, \quad j \geq k+1.$ then we have the factorization property:

$$T(A_1(x_1), \ldots, A_n(x_n)) = T(A_1(x_1), \ldots, A_k(x_k))\ \ T(A_{k+1}(x_{k+1}), \ldots, A_n(x_n)); \tag{2.4}$$

- Unitarity: We define the *anti-chronological products* using a convenient notation introduced by Epstein-Glaser, adapted to the Grassmann context. If $X = \{j_1, \ldots, j_s\} \subset N \equiv \{1, \ldots, n\}$ is an ordered subset, we define

$$T(X) \equiv T(A_{j_1}(x_{j_1}), \ldots, A_{j_s}(x_{j_s})). \tag{2.5}$$

  Let us consider some Grassmann variables θ_j, of parity f_j, $j = 1, \ldots, n$ and let us define

$$\theta_X \equiv \theta_{j_1} \cdots \theta_{j_s}. \tag{2.6}$$

  Now let $(X_1, \ldots, X_r)$ be a partition of $N = \{1, \ldots, n\}$ where $X_1, \ldots, X_r$ are ordered sets. Then we define the sign $\epsilon(X_1, \ldots, X_r)$ through the relation

$$\theta_1 \cdots \theta_n = \epsilon(X_1, \ldots, X_r)\ \theta_{X_1} \ldots \theta_{X_r} \tag{2.7}$$

Then the antichronological products according to

$$(-1)^n \bar{T}(N) \equiv \sum_{r=1}^{n} (-1)^r \sum_{I_1,\dots,I_r \in Part(N)} \epsilon(X_1,\dots,X_r)\; T(X_1)\cdots T(X_r) \tag{2.8}$$

Then the unitarity axiom is:

$$\bar{T}(N) = T(N)^\dagger. \tag{2.9}$$

- The "initial condition"

$$T(A(x)) = A(x). \tag{2.10}$$

It can be proved that this system of axioms can be supplemented with

$$T(A_1(x_1),\dots,A_n(x_n)) = \sum \epsilon \quad <\Omega, T(A'_1(x_1),\dots,A'_n(x_n))\Omega> \quad :A''_1(x_1),\dots,A''_n(x_n): \tag{2.11}$$

where A'_i and A''_i are Wick submonomials of A_i such that $A_i =: A'_i A''_i :$ and the sign ϵ takes care of the permutation of the Fermi fields; here Ω is the vacuum state. This is called the *Wick expansion property*.

We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated to arbitrary Wick monomials $A_1,\dots,A_n$; explicitly:

$$\omega(<\Omega, T^{A_1,\dots,A_n}(X)\Omega>) \leq \sum_{l=1}^{n} \omega(A_l) - 4(n-1) \tag{2.12}$$

where by $\omega(d)$ we mean the order of singularity of the (numerical) distribution d and by $\omega(A)$ we mean the canonical dimension of the Wick monomial W.

Up to now, we have defined the chronological products only for self-adjoint Wick monomials $W_1,\dots,W_n$ but we can extend the definition for Wick polynomials by linearity.

The construction of Epstein-Glaser is based on the following recursive procedure. Suppose that we know the chronological products up to order $n-1$. Then we define the following expression:

$$D(N) \equiv - \sum_{(X,Y)\in Part(N)} (-1)^{|Y|}\; \epsilon(X,Y)\; [\bar{T}(X), T(Y)] \tag{2.13}$$

where the partitions (X,Y) are restricted by $n \in X, Y \neq \emptyset$, $|Y|$ is the cardinal of Y and the commutator is graded. These restrictions guarantee that $|X|,|Y| < n$ so the expressions in the right-hand side of the previous expression are known by the induction hypothesis. Then it can be proved that the expression $D(N) = D(A_1(x_1),\dots,A_n(x_n))$ has causal support in the variables $x_1 - x_n,\dots,x_{n-1} - x_n$; accordingly is called the *causal commutator*. One can causally split $D(N)$ as

$$D(N) = D^{\rm adv}(N) - D^{\rm ret}(N) \tag{2.14}$$

with $D^{\rm adv}(N)$ (resp. $D^{\rm ret}(N)$) with support in the upper (resp. lower) light cone. From these expression one can construct the chronological products $T(N)$ in order n in a standard way.

3 Trivial Lagrangians

Here we have proved the following

Theorem 3.1 *Suppose the chronological products are chosen such that we have gauge invariance (1.9) up to order n and we modify the first order chronological products (the interaction Lagrangian) by a coboundary (a trivial Lagrangian):*

$$T^I \to T^I + T^I_0 \tag{3.1}$$

where

$$T^I_0 \equiv \bar{s} B^I \tag{3.2}$$

is a coboundary. Then the chronological products, up to order n get modified by a coboundary also:

$$T(T^{I_1}(x_1), \ldots, T^{I_n}(x_n)) \to T(T^{I_1}(x_1), \ldots, T^{I_n}(x_n)) + T_0(T^{I_1}(x_1), \ldots, T^{I_n}(x_n)) \tag{3.3}$$

where the expression $T_0(T^{I_1}(x_1), \ldots, T^{I_n}(x_n))$ is a coboundary.

Proof: Let us work first in the **second order** of the perturbation theory. From (2.13) we see that the second-order causal commutator coincides with the usual commutator

$$D(A(x), B(y)) = [A(x), B(y)]. \tag{3.4}$$

Suppose we make the redefinition (3.1); then we have

$$T(T^I(x), T^J(y)) \to T(T^I(x), T^J(y)) + T_0(T^I(x), T^J(y)) \tag{3.5}$$

where

$$T_0(T^I(x), T^J(y)) = T(T^I_0(x), T^J(y)) + T(T^I(x), T^J_0(y)) + T(T^I_0(x), T^J_0(y)). \tag{3.6}$$

We easily determine by direct computations that

$$D(T^I_0(x), T^J(y)) + D(T^I(x), T^J_0(y)) = \bar{s} B_0(T^I(x), T^J(y)) \tag{3.7}$$

where

$$B_0(T^I(x), T^J(y)) \equiv D(B^I(x), T^J(y)) + (-1)^{|I|}\ D(T^I(x), B^J(y)) \tag{3.8}$$

To see how this works let us compute

$$\begin{aligned} D(T^I_0(x), T^J(y)) = D(\bar{s} B^I(x), T^J(y)) = [\bar{s} B^I(x), T^J(y)] \\ = [d_Q B^I(x) + i \partial_\mu B^{I\mu}(x), T^J(y)] \\ = d_Q\ [B^I(x), T^J(y)] +\ (-1)^{|I|}\ [B^I(x), d_Q T^J(y)] + i \frac{\partial}{\partial x^\mu} [B^{I\mu}(x), T^J(y)] \end{aligned} \tag{3.9}$$

where we have used the fact that d_Q verifies the (graded) Leibniz rule. In the second term above we use first-order gauge invariance (1.8) and obtain

$$D(T_0^I(x), T^J(y)) =$$
$$= d_Q \; [B^I(x), T^J(y)] + \; i \; (-1)^{|I|} \; \frac{\partial}{\partial y^\mu} \; [B^I(x), T^J(y)] + i \frac{\partial}{\partial x^\mu} [B^{I\mu}(x), T^J(y)]. \tag{3.10}$$

The second term of left hand side of (3.7) is computed in the same way and regrouping the terms we get the result.

Now we see that in (3.7) both sides have causal support, so the causal splitting produces

$$T(T_0^I(x), T^J(y)) + T(T^I(x), T_0^J(y)) = \bar{s} B_0^F(T^I(x), T^J(y)) \tag{3.11}$$

where

$$B_0^F(T^I(x), T^J(y)) \equiv T(B^I(x), T^J(y)) + (-1)^{|I|} \; T(T^I(x), B^J(y)) \tag{3.12}$$

This means that the first two terms from the right-hand side of (3.6) are a coboundary. Because we have $sT_0^I = s\bar{s}B^I = 0$ according to (1.17) it follows that we can repeat the computations leading to (3.7) + (3.8) with $T^I \rightarrow T_0^I$ and we obtain instead of (3.11)

$$T(T_0^I(x), T_0^J(y)) = \frac{1}{2} \; \bar{s} B_0^F(T_0^I(x), T_0^J(y)) \tag{3.13}$$

so the last term of (3.6) is a coboundary. In conclusion we have the desired property in the second-order of perturbation theory:

$$T_0(T^I(x), T^J(y)) = \bar{s} B^F(T^I(x), T^J(y)) \tag{3.14}$$

where

$$B^F(T^I(x), T^J(y)) = B_0^F(T^I(x), T^J(y)) + \frac{1}{2} \; B_0^F(T_0^I(x), T_0^J(y)). \tag{3.15}$$

We have proved that if we modify the interaction Lagrangian T^I by a trivial Lagrangian (a coboundary), then the second order chronological products get modified by a coboundary also.

(ii) It is illuminating to push the proof to the **third** order of the perturbation theory. We suppose that we have fixed the second-order chronological products such that we have gauge invariance in the second-order (1.9) for $n = 2$. From (2.13) we have similarly with (3.4):

$$\begin{aligned} D(A(x), B(y), C(z)) = -[\bar{T}(A(x), B(y)), C(z)] \\ -(-1)^{|B||C|}\ [T(A(x), C(z)), B(y)] - (-1)^{|A|(|B|+|C|)}\ [T(B(y), C(z)), A(x)]. \end{aligned} \tag{3.16}$$

Also, similarly to (3.6), we have

$$\begin{aligned} T_0(T^I(x), T^J(y), T^K(z)) = \\ T(T_0^I(x), T^J(y), T^K(z)) + T(T^I(x), T_0^J(y), T^K(z)) + T(T^I(x), T^J(y), T_0^K(z)) \\ +T(T^I(x), T_0^J(y), T_0^K(z)) + T(T_0^I(x), T^J(y), T_0^K(z)) + T(T_0^I(x), T_0^J(y), T^K(z)) \\ +T_0(T_0^I(x), T_0^J(y), T_0^K(z)). \end{aligned} \tag{3.17}$$

Guided by the previous (second-order) analysis we prove by direct computation that

$$\begin{aligned} D(T_0^I(x), T^J(y), T^K(z)) + D(T^I(x), T_0^J(y), T^K(z)) + D(T^I(x), T^J(y), T_0^K(z)) \\ = \bar{s}B_0(T^I(x), T^J(y), T^K(z)) \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} B_0(T^I(x), T^J(y), T^K(z)) = D(B^I(x), T^J(y), T^K(z)) \\ +(-1)^{|I|}\ D(T^I(x), B^J(y), T^K(z)) + (-1)^{|I|+|J|}\ D(T^I(x), T^J(y), B^K(z)). \end{aligned} \tag{3.19}$$

In this proof gauge invariance in the second-order must be used as in (3.9) $\Rightarrow$ (3.10) above. Now both hand sides of (3.18) are with causal support, so the causal splitting gives

$$\begin{aligned} T(T_0^I(x), T^J(y), T^K(z)) + T(T^I(x), T_0^J(y), T^K(z)) + T(T^I(x), T^J(y), T_0^K(z)) \\ = \bar{s}B_0^F(T^I(x), T^J(y), T^K(z)) \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} B_0^F(T^I(x), T^J(y), T^K(z)) = T(B^I(x), T^J(y), T^K(z)) \\ +(-1)^{|I|}\ T(T^I(x), B^J(y), T^K(z)) + (-1)^{|I|+|J|}\ T(T^I(x), T^J(y), B^K(z)). \end{aligned} \tag{3.21}$$

The last two terms of (3.17) can be easily computed: Using gauge invariance in the second order and (3.11) + (3.13) we see that the expression

$$T(T^I(x) + \alpha T_0^I(x), T^J(y) + \alpha T_0^J(y))$$

is gauge invariant for an arbitrary $\alpha \in \mathbb{R}$ so the previous proof of (3.20) + (3.21) with stays true for $T^I \to T^I + \alpha T_0^I$. The coefficients of α^2 and α^3 give the coboundary structure of the last two terms of (3.17) and we have the result for $n = 3$.

(iii) Finally we go to the general case of arbitrary n. We want to determine the expression

$$T_0(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) \equiv$$
$$T(T^{I_1}(x_1)+T_0^{I_1}(x_1),\ldots,T^{I_n}(x_n)+T_0^{I_n}(x_n)) - T(T^{I_1}(x_1,\ldots,T^{I_n}(x_n)) \tag{3.22}$$

and prove that it is a coboundary. We introduce the following notations:

$$T_{l_1,\ldots,l_r}(T^{I_1}(x_1),\ldots,T^{I_n}(x_n))$$

is obtained from

$$T(T^{I_1}(x_1),\ldots,T^{I_n}(x_n))$$

by the substitutions

$$T^{I_{l_1}} \to T_0^{I_{l_1}},\ldots,T^{I_{l_r}} \to T_0^{I_{l_r}}$$

so it easily follows that

$$T_0(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) = \sum_{r=1}^{n} \sum_{l_1<\cdots<l_r} T_{l_1,\ldots,l_r}(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)). \tag{3.23}$$

We will prove that all the expressions

$$T_{l_1,\ldots,l_r}(T^{I_1}(x_1),\ldots,T^{I_n}(x_n))$$

are coboundaries. As in the cases $n = 2, 3$ from above, we first prove a generalization of (3.11) + (3.12) and (3.20) + (3.21) by induction. More precisely, we suppose that we have fixed gauge invariance (1.9) up to the order $n-1$ and proved

$$\sum_{l=1}^{p} T(T^{I_1}(x_1),\ldots,T_0^{I_l}(x_l),\ldots,T^{I_n}(x_n)) = \bar{s}B_0^F(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) \tag{3.24}$$

where

$$B_0^F(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) = \sum_{l=1}^{p} \prod_{j<l} (-1)^{f_j}\; T(T^{I_1}(x_1),\ldots,B^{I_l}(x_l),\ldots,T^{I_n}(x_n)) \tag{3.25}$$

for $p = 1,\ldots,n-1$. We want to prove the same result for $p = n$. We determine the sum of causal commutators

$$D_n \equiv \sum_{l=1}^{p} D(T^{I_1}(x_1),\ldots,T_0^{I_l}(x_l),\ldots,T^{I_n}(x_n)) \tag{3.26}$$

using the definition (2.13) with the substitution $T^{I_l} \to T_0^{I_l}$; we have two type of terms: with $l \in X$ and with $l \in Y$

$$D_n = -\sum_{l=1}^{p} \Big(\sum_{(X,Y)\in Part(N), l\in X} (-1)^{|Y|}\; \epsilon(X,Y)\; [\bar{T}_{sB_l}(X), T(Y)]$$
$$+ \sum_{(X,Y)\in Part(N), l\in Y} (-1)^{|Y|}\; \epsilon(X,Y)\; [\bar{T}(X), T_{sB_l}(Y)]\Big) \tag{3.27}$$

where the expression $\bar{T}_{sB_l}(X), T_{sB_l}(Y)$ are obtained from $\bar{T}(X), T(Y)$ with the substitution $T^{I_l} \to \bar{s}B^{I_l}$. We invert the order of summation and obtain

$$D_n = -\sum_{(X,Y)\in Part(N)} (-1)^{|Y|}\ \epsilon(X,Y)\ [\sum_{l\in X} \bar{T}_{sB_l}(X), T(Y)]$$
$$- \sum_{(X,Y)\in Part(N)} (-1)^{|Y|}\ \epsilon(X,Y)\ [\bar{T}(X), \sum_{l\in Y} T_{sB_l}(Y)]. \tag{3.28}$$

Now, the restrictions $n \in X, Y \neq \emptyset$ from the definition of the causal commutator implies that $|X|, |Y| < n$ so we can apply the induction hypothesis and get

$$D_n = -\sum_{(X,Y)\in Part(N)} (-1)^{|Y|}\ \epsilon(X,Y)\ [\bar{s}\bar{B}^F_0(X), T(Y)]$$
$$- \sum_{(X,Y)\in Part(N)} (-1)^{|Y|}\ \epsilon(X,Y)\ [\bar{T}(X), \bar{s}B^F_0(Y)]. \tag{3.29}$$

We compute the two commutators as before; for instance

$$[\bar{s}\bar{B}^F_0(X), T(Y)] = [d_Q\bar{B}^F_0(X) + i\ \delta_X\bar{B}^F_0(X), T(Y)]$$
$$= d_Q[\bar{B}^F_0(X), T(Y)] + (-1)^{\phi_X}\ [\bar{B}^F_0(X), d_QT(Y)] + i\ \delta_X[\bar{B}^F_0(X), T(Y)]$$

where the operator δ_X is the operator (1.11) applied to a cocycle depending only on the variables $x_j,\ j \in X$ and

$$\phi_X \equiv \sum_{j\in X}\ f_j \tag{3.30}$$

is the Fermi number of $T(X)$. Now we apply the gauge invariance induction hypothesis to express $d_QT(Y)$ as $i\ \delta_YT(Y)$ and finally

$$[\bar{s}\bar{B}^F_0(X), T(Y)] = \bar{s}[\bar{B}^F_0(X), T(Y)]. \tag{3.31}$$

We do the same type of computation for the second commutator from (3.29) and end up with

$$D_n = \bar{s}B \tag{3.32}$$

where

$$B = -\sum_{(X,Y)\in Part(N)} (-1)^{|Y|}\ \epsilon(X,Y)\ [\bar{B}^F_0(X), T(Y)]$$
$$- \sum_{(X,Y)\in Part(N)} (-1)^{|Y|}\ (-1)^{\phi_X}\ \epsilon(X,Y)\ [\bar{T}(X), B^F_0(Y)]. \tag{3.33}$$

The previous expression can be rewritten using the induction hypothesis under the form

$$B^F_0(X) = \sum_{l\in X}\ (-1)^{\phi_{X,l}}\ T_{B_l}(X), \qquad |X| < n \tag{3.34}$$

where

$$\phi_{X,l} \equiv \sum_{j\in X, l<l} f_j \tag{3.35}$$

(remember that X is an ordered set) and $T_{B_l}(X)$ is obtained from $T(X)$ with the substitution $T^{I_l} \to B^{I_l}$; a similar formula is true for $\bar{B}^F_0$.

We substitute in (3.33), invert the order of summation and dealing carefully with the signs we obtain

$$B = \sum_{l=1}^n \prod_{j<l} (-1)^{f_j}\ D(T^{I_1}(x_1),\ldots,B^{I_l}(x_l),\ldots,T^{I_n}(x_n)). \tag{3.36}$$

It follows that in (3.32) both sides are causal expressions, so the causal splitting gives (3.24) + (3.25) for $p = n$.

To prove that the expression T_0 from (3.22) is a coboundary we proceed as follows. We replace the induction hypothesis (3.24) + (3.25) by a stronger induction hypothesis, namely we suppose that we have for $p = 1,\ldots,n-1$ and $r < p$ the following relation

$$\sum_{l_1<\ldots<l_r} T_{l_1,\ldots,l_r}(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) = \bar{s}B^F_{r-1}(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) \tag{3.37}$$

where

$$\begin{aligned} B^F_r(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) &= \frac{1}{r+1}\sum_{s=1}^p \prod_{j<s} (-1)^{f_j}\ \times \\ &\times \sum_{l_1<\ldots<l_r} T_{l_1,\ldots,l_r}(T^{I_1}(x_1),\ldots,B^{I_l}(x_l),\ldots,T^{I_n}(x_n)) \end{aligned} \tag{3.38}$$

where in the sum over $l_1 < \ldots < l_r$ we impose $\{l_1,\ldots,l_r\} \cap \{s\} = \emptyset$.

In this case we can easily prove that the expressions

$$T(T^{I_1}(x_1) + \alpha\ T^{I_1}_0(x_1),\ldots,T^{I_n}(x_n) + \alpha\ T^{I_p}_0(x_p)),\ \ p < n$$

are gauge invariant in the sense (1.9) so we can reconsider the proof of (3.24) + (3.25) for $p = n$ with $T^I \to T^I + \alpha\ T^I_0$ for an arbitrary $\alpha \in \mathbb{R}$. All expressions are polynomials in α and the coefficient of α^r gives exactly the relations (3.37) + (3.38) and this finishes the induction. Now we have from (3.23)

$$T_0(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) = \bar{s}B^F(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) \tag{3.39}$$

i.e. a coboundary, where

$$B^F = \sum_{r=0}^{n-1} B^F_r \tag{3.40}$$

and this finishes the proof. ■

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